Published by Institute of Physics Publishing for SISSA/ISAS
Received: *January 24, 2002*
Revised: *April 22, 2002*
Accepted: *April 30, 2002*

Towards the realistic fermion masses with a single family in extra dimensions

Maxim Libanov and Emin Nougaev

Institute for Nuclear Research of the Russian Academy of Sciences
60th October Anniversary Prospect 7a, 117312, Moscow, Russia
E-mail: ml@ms2.inr.ac.ru, emin@ms2.inr.ac.ru

Abstract: In a class of multidimensional models, topology of a thick brane provides three chiral fermionic families with hierarchical masses and mixings in the effective four-dimensional theory, while the full model contains a single vector-like generation. We carry out numerical simulations and reproduce all known Standard Model fermion masses and mixings in one of these models.

Keywords: Extra Large Dimensions, Quark Masses and SM Parameters, Beyond Standard Model.

JHEP04(2002)055

Contents

1. Introduction **1**

2. The setup **2**

3. The vortex background **4**

4. Mass hierarchy: cases of global and gauged $\mathrm{U}_g(1)$ **6**

4.1 Global $\mathrm{U}_g(1)$ 6

4.2 Gauged $\mathrm{U}_g(1)$ 9

5. Results for Standard Model fermions **10**

6. Conclusions **12**

1. Introduction

One of interesting features inherent in the theories with more than four spacetime dimensions is a possibility to explain the mysterious pattern of fermion mass hierarchies [1, 2, 3] (see also review [4] and references therein). In the previous series of works [2, 3, 5], we have constructed a model in which a single family of fermions, with vector-like couplings to the Standard Model (SM) gauge groups in 6 dimensions, gives rise to three generations of chiral Standard Model fermions in 4 dimensions. This mechanism is based on localization of fermion zero modes on a two-dimensional vortex with winding number n. In particular, in the paper [2] the case of global vortex (i.e. group of symmetry of the vortex $\mathrm{U}_g(1)$ is global) with winding number $n = 3$ was considered. In this setup three localized fermionic zero modes appear due to specific Yukawa coupling to the vortex scalar Φ. Coupling of fermions to the SM Higgs doublet H results in four-dimensional effective fermion masses. Inter-generation mixings occur due to explicit breaking of $\mathrm{U}_g(1)$ symmetry. The latter point is the main drawback of this model since it does not allow to gauge $\mathrm{U}_g(1)$ symmetry. In the paper [3] this problem was overcome. The price for unbroken $\mathrm{U}_g(1)$, however, is the necessity to invoke higher dimension operators in the scalar-fermion interactions.[1] More precisely, to obtain n fermionic generations we consider the vortex with winding number 1 but fermions coupled to the vortex scalar raised to the nth power. In both cases (global

[1]The requirement of renormalizability of the theory does not make much sense anyway in the six-dimensional models, since even usual Yukawa scalar-fermion-fermion coupling is non-renormalizable.

and gauged $\mathrm{U}_g(1)$ symmetry) the hierarchical pattern of the fermion masses occurs due to different profiles of the fermionic wave functions in the transverse extra dimensions. Namely, in refs. [2, 3] a crude dimensional analysis has been presented which shows that hierarchical mass pattern has the form

$$1 : g^2 : g^4 \,, \tag{1.1}$$

where g is small Yukawa coupling, which yields the localization of fermionic modes. Also, in refs. [2, 3] it has been pointed out that any exact prediction of the fermion masses (as well as a check of the eq. (1.1)) can be obtained only numerically.

In this paper we return to the models of refs. [2, 3] and pay special attention to a check of the eq. (1.1). We will see that this equation literally does not hold. Namely, in the case of gauged vortex the hierarchical mass pattern takes the form

$$1 : g : g^2 \,.$$

At the same time, in the case of global vortex unnatural hierarchical pattern appears. We will also evaluate fermion masses numerically. For reasonable values of parameters in the case of the gauged vortex our results reproduce SM fermionic mass pattern. Furthermore, we will obtain Cabbibo-Kobayashi-Maskawa (CKM) matrix for the same set of parameters, and see that six-dimensional theory reproduces the mixing matrix quite well.

2. The setup

In what follows we restrict ourselves to the model of ref. [3]. In this section we give a brief description of this construction. Our notations coincide with those used in refs. [2, 3]. In particular, six-dimensional coordinates x_A are labeled by capital Latin indices $A, B = 0, \ldots, 5$. Four-dimensional coordinates x_μ are labeled by Greek indices $\mu, \nu = 0, \ldots, 3$. The Minkowski metric is $g_{AB} = \mathrm{diag}(+, -, \ldots, -)$. We use also chiral representation for six-dimensional Dirac Γ-matrices (see ref. [2]). Besides this we introduce polar coordinates r, and θ in the x_4, x_5 plane.

The matter field content of the six-dimensional theory is summarized in table 1. There are three scalar fields. One of these Φ, together with the $\mathrm{U}_g(1)$ gauge field, forms a vortex, while two other scalars, X and H, develop profiles localized on the vortex. The potential term of the lagrangian which gives rise to the non-trivial profiles for the scalar fields has the following form

$$V_s = \frac{\lambda}{2}\left(|\Phi|^2 - v^2\right)^2 + \frac{\kappa}{2}\left(|H|^2 - \mu^2\right)^2 + h^2|H|^2|\Phi|^2 + \frac{\rho}{2}\left(|X|^2 - v_1^2\right)^2 + \eta^2|X|^2|\Phi|^2 \,. \tag{2.1}$$

There is also one fermionic generation which consists of five six-dimensional fermions Q, D, U, L and E. These fermions have vector-like coupling to SM gauge bosons, (a subtle issue of localization of the latters we do not discuss here), and axial couplings with the

fields	profiles	charges		representations	
		$\mathrm{U}_g(1)$	$\mathrm{U}_Y(1)$	$\mathrm{SU}_W(2)$	$\mathrm{SU}_C(3)$
scalar Φ	$F(r)\mathrm{e}^{i\theta}$ $F(0)=0,\, F(\infty)=v$	$+1$	0	**1**	**1**
scalar X	$X(r)$ $X(0)=v_X,\, X(\infty)=0$	$+1$	0	**1**	**1**
scalar H	$H(r)$ $H(0)=v_H,\, H(\infty)=0$	-1	$+1/2$	**2**	**1**
fermion Q	3 L zero modes	axial $+3/2$	$+1/6$	**2**	**3**
fermion U	3 R zero modes	axial $-3/2$	$+2/3$	**1**	**3**
fermion D	3 R zero modes	axial $-3/2$	$-1/3$	**1**	**3**
fermion L	3 L zero modes	axial $+3/2$	$-1/2$	**2**	**1**
fermion E	3 R zero modes	axial $-3/2$	-1	**1**	**1**

Table 1: Scalars and fermions with their gauge quantum numbers [3]. For convenience, we describe here also the profiles of the classical scalar fields and fermionic wave functions in extra dimensions.

vortex background[2]

$$V_1 = g_q\Phi^3\bar{Q}\frac{1-\Gamma_7}{2}Q + g_u\Phi^{*3}\bar{U}\frac{1-\Gamma_7}{2}U + g_d\Phi^{*3}\bar{D}\frac{1-\Gamma_7}{2}D + \\ + g_l\Phi^3\bar{L}\frac{1-\Gamma_7}{2}L + g_e\Phi^{*3}\bar{E}\frac{1-\Gamma_7}{2}E + \text{h.c.} \qquad (2.2)$$

(Γ_7 is a six-dimensional analog of four-dimensional matrix γ_5).

Note that the scalar background Φ^3, where the field Φ has the winding number one, $\Phi = F(r)\mathrm{e}^{i\theta}$, has exactly the same topological properties as the background Φ_1 of the vortex with winding number three, $\Phi_1 = F_1(r)\mathrm{e}^{3i\theta}$. As a result, the interactions eq. (2.2) give rise to *three* left-handed (right-handed) zero modes for each of the fermions Q, L (U, D, E). All of these modes are localized in the core of the vortex,[3] and could be identified with the three generations of the chiral SM fermions. The last point, however, is not quite fair since these *zero* modes are *massless* from the four-dimensional point of view. To give small (compared to the vortex scale $\sqrt{\lambda}v$) masses to these modes we introduce the following interactions of the fermions with the Higgs doublet H and singlet X

$$V_2 = Y_d HX\bar{Q}\frac{1-\Gamma_7}{2}D + Y_u\tilde{H}X^*\bar{Q}\frac{1-\Gamma_7}{2}U + Y_l HX\bar{L}\frac{1-\Gamma_7}{2}E + \text{h.c.}, \qquad (2.3)$$

[2]For the scalar-fermion interactions, we take the most general operators of the lowest order, consistent with gauge invariance. On the other hand, in the scalar self-interaction (2.1) there could be terms of the form ΦX^* or $\Phi X^*|H|^2$ which we do not include. These terms, as well as terms of the form $\Phi^2 X\bar{Q}(1-\Gamma_7)Q$ in the (2.2), can be forbidden, for example, by the imposition of Z_4 discrete symmetry: $\Phi \to i\Phi$, $Q \to \exp(i\Gamma_7\frac{3\pi}{4})Q$, $X \to X$. This symmetry is explisitly broken by the interaction (2.3), (2.4), and, therefore the undesirable terms will be generated by the radiative corrections. However, the value of these corrections is of order $\frac{Y^2}{(4\pi)^3}\epsilon\Lambda^4$ (for the operator $|H|^2\Phi X^*$, for instance; Λ is the cut off), and for our choice of the parameters (see section 5) is of order $10^{-6}(\Lambda/M)^4\lambda$, where M is mass scale, and $\lambda = 1/M^2$. So, if we choose the cut off to be of order $\Lambda \sim 10M$, then this corrections can be neglected.

[3]These modes were discussed in detail in ref. [2].

where $\tilde{H}_\alpha = \epsilon_{\alpha\beta} H^*_\beta$, α, β are $\mathrm{SU}_W(2)$ indices. Finally, there is one more gauge invariant in the scalar-fermions interaction

$$V_3 = Y_d \epsilon_d H \Phi \bar{Q} \frac{1-\Gamma_7}{2} D + Y_u \epsilon_u \tilde{H} \Phi^* \bar{Q} \frac{1-\Gamma_7}{2} U + Y_l \epsilon_l H \Phi \bar{L} \frac{1-\Gamma_7}{2} E + \text{h.c.}, \tag{2.4}$$

(we denote Yukawa coupling constants in eq. (2.3) as $Y_{u,d,l}$, and in eq. (2.4) as $Y_u\epsilon_u, \dots$ for convenience). This term yields inter-generation mixings.

It is worth noting that to obtain small fermion masses and mixing, it is not necessary to take Y or ϵ small. Small masses and mixings originate from small overlaps of the fermionic wave functions and the scalar profiles.

Now let us consider this model in more detail. First of all we will present stable non-trivial configuration for the scalars and $\mathrm{U}_g(1)$ gauge fields. Then we will find fermion zero modes in this background. Finally, we will calculate the fermion masses and the CKM matrix.

3. The vortex background

As described in the previous section, in our model there are three scalar fields and one $\mathrm{U}_g(1)$ gauge field which form a non-trivial vortex background. In this section we present a stable solution of the static field equations which meets all the requirements.

The set of the static equations to the background has the following form (prime denotes the derivative with respect to r; e is $\mathrm{U}_g(1)$ gauge charge),

$$\begin{aligned}
F'' + \frac{1}{r}F' - \frac{1}{r^2}F(1-A)^2 - \lambda v^2 F\left(F^2-1\right) - h^2 F H^2 - \eta^2 F X^2 &= 0\,,\\
A'' - \frac{1}{r}A' + 2e^2v^2F^2(1-A) - 2e^2H^2A - 2e^2X^2A &= 0\,,\\
H'' + \frac{1}{r}H' - \frac{1}{r^2}HA^2 - h^2Hv^2F^2 - \kappa H(H^2-\mu^2) &= 0\,,\\
X'' + \frac{1}{r}X' - \frac{1}{r^2}XA^2 - \eta^2Xv^2F^2 - \rho X(X^2-{v_1}^2) &= 0\,,
\end{aligned} \tag{3.1}$$

with the boundary conditions

$$\begin{aligned}
&F(r)=0\,, \quad H'(r)=0\,, \quad X'(r)=0\,, \quad A(r)=0\,, \qquad \text{at } r=0\,,\\
&F(r)\to 1\,, \quad H(r)\to 0\,, \quad X(r)\to 0\,, \quad A(r)\to 1\,, \qquad \text{at } r\to\infty\,.
\end{aligned} \tag{3.2}$$

Here we use the standard anzatz for Abrikosov-Nielsen-Olesen vortex ($i,j=4,5$)

$$\begin{aligned}
A_i(x) &= -\frac{1}{er^2}\epsilon_{ij}x_jA(r)\,, \qquad & \Phi(x) &= vF(r)e^{i\theta}\,,\\
H_\alpha(x) &= \delta_{2\alpha}H(r)\,, & X(x) &= X(r)\,.
\end{aligned} \tag{3.3}$$

Let us briefly discuss the choice of the boundary conditions (3.2) and the anzatz (3.3). At infinity the fields should tend to their vacuum expectation values (VEV) which are determined from the potential (2.1). In general at arbitrary set of the coupling constants the potential (2.1) may admit several minima including those in which X and (or) H are non-zero. However, there are several regions in the space of the parameters where the potential (2.1) has *only one* minimum $\Phi = v$, $H = X = 0$. In particular, one of the regions is determined by the conditions $h^4 > \lambda\kappa$, $\lambda\rho > \eta^4$, $\eta^2 v^2 > \rho v_1^2$, and in what follows we will assume that these conditions are satisfied.

In general, any configuration of the fields Φ, H, and X may be characterized by three topological charges $Q_T = (Q_T^\Phi, Q_T^H, Q_T^X)$. The zero VEVs of X and H admit the existence of the topologically trivial configuration of these fields (3.3) (in spite of the interaction with winded gauge field), $Q_T = (1, 0, 0)$. In principle, there may exist a solution of the field equation with $Q_T = (1, -1, 1)$ and with the same topological properties of the gauge field A_μ. However, as we will see, the solution with $Q_T = (1, 0, 0)$ is stable, and, therefore, the choice of the anzatz (3.3) is apropriate.

As mentioned in the previous section, scalar-fermion interactions (2.3), (2.4) generate low energy mass terms for fermionic modes. Thus, to obtain non-zero fermion masses one should have a non-trivial solution for $H(r)$ and $X(r)$. On the other hand, the boundary value problem (3.1), (3.2) always adopts trivial solutions $H = 0$ or (and) $X = 0$. Fortunately, it has been pointed out in [6], that in some region of parameters these trivial solutions are unstable. Namely, the necessary conditions to have non-trivial stable solutions are $h^2 v^2 > \kappa\mu^2$ and $\eta^2 v^2 > \rho v_1^2$ [6]. However, it turns out that these conditions are insufficient. To clarify the situation, let us study the problem of the stability in more detail. The stability of the scalar Φ and gauge fields is guaranteed by topology. To investigate the stability of the static solution $H(r)$ one linearizes equations of motion (3.1) and obtain Schrödinger-type problem with the potential

$$h^2 v^2 F^2(r) + \frac{1}{r^2} A^2 - \kappa(\mu^2 - 3H^2(r)) \,. \tag{3.4}$$

At large r the solution of eq. (3.1), $H(r)$, has asymptotic $H \sim \exp(-\sqrt{h^2 v^2 - \kappa\mu^2}\, r)$. So, at large enough h the Higgs profile becomes narrow compared to $F(r)$. This means, in particular, that there is a region of r where $H(r)$ becomes negligible while $A(r)$ and $F(r)$ are still almost equal to zero. Therefore, the main contribution to the potential (3.4) in this region comes from r-independent term $-\kappa\mu^2$ and is negative. Thus, the potential becomes not positively definite, and there may exist negative levels which mean the instability. The numerical calculations show that this is the case.

To be specific, we have chosen the following set of the parameters: $v = \mu = 1 \cdot M^2$, $v_1 = 0.3 \cdot M^2$, $\lambda = \kappa = \rho = 1 \cdot M^{-2}$, $h = 1.02 \cdot M^{-1}$, $\eta = 0.31 \cdot M^{-1}$, $e = 0.04 \cdot M^{-1}$, where M is a unit of mass which we will define in section (5). In numerical calculations we assume $M = 1$. To solve the boundary value problem (3.1), (3.2) we have used the relaxation method [7]. The solutions for profiles $F(r)$, $H(r)$, $X(r)$ and $A(r)$ in the vicinity of the origin are presented in figure 1. The potential (3.4) for these solutions is positive and, therefore, the solutions are stable.

The vortex background in the case of global $\mathrm{U}_g(1)$ group can be obtained from the eqs. (3.1), (3.2) by the setting $e = 0$, $A(r) = 0$ ($A(\infty) = 0$). The profiles for the scalar fields have the same shapes as in the gauge case. We do not present the corresponding plot here.

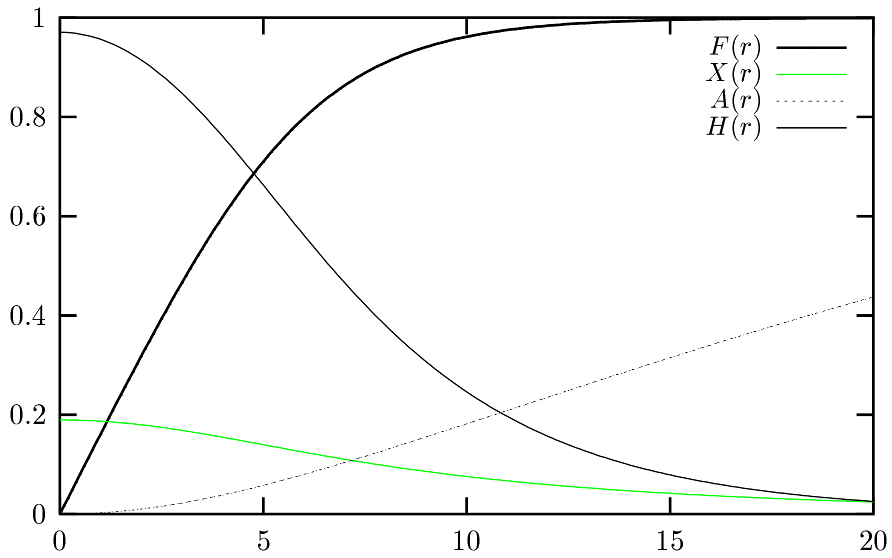

Figure 1: Profiles of the background at $\lambda = \kappa = \rho = 1$, $v = \mu = 1$, $e = 0.04$, $h = 1.02$, $\eta = 0.31$, $v_1 = 0.3$.

4. Mass hierarchy: cases of global and gauged $\mathrm{U}_g(1)$

Now we have all ingredients to find fermionic zero modes in the vortex background and to obtain hierarchy of the low energy fermionic masses. To do this let us first briefly remind the analysis given in refs. [2, 3].

4.1 Global $\mathrm{U}_g(1)$

Let us consider first the case of global $\mathrm{U}_g(1)$ group and one fermionic family, say Q and U. The part of the lagrangian describing the interaction of six-dimensional spinor Q with vortex scalar Φ is

$$L_Q = i\bar{Q}\Gamma^A\partial_A Q - \left(g_q\Phi^3\bar{Q}\frac{1-\Gamma_7}{2}Q + \text{h.c.}\right). \tag{4.1}$$

This lagrangian is invariant under $\mathrm{U}_g(1)$ rotations

$$Q \to \mathrm{e}^{i\frac{3\alpha}{2}\Gamma_7}Q\,, \qquad \Phi \to \mathrm{e}^{i\alpha}\Phi\,.$$

In the vortex background, one can expand an arbitrary spinor in eigenfunctions of the corresponding Dirac operator. There is a set of massive heavy modes (with masses of order gv^3) and zero modes. The careful analysis given in [8, 2, 3] shows that in the vortex background with winding number one there are exactly three localized zero modes describing four-dimensional left-handed spinors. To obtain three right-handed zero modes one should consider a six-dimensional spinor U which has an axial charge $-3/2$ under $\mathrm{U}_g(1)$ with the lagrangian

$$L_U = i\bar{U}\Gamma^A\partial_A U - \left(g_u\Phi^3\bar{U}\frac{1+\Gamma_7}{2}U + \text{h.c.}\right). \tag{4.2}$$

These left-handed and right-handed zero modes have the form

$$Q_p = \begin{pmatrix} 0 \\ q_p(r)\mathrm{e}^{i(3-p)\theta} \\ q_{4-p}(r)\mathrm{e}^{i(1-p)\theta} \\ 0 \end{pmatrix}, \qquad U_p = \begin{pmatrix} u_{4-p}(r)\mathrm{e}^{i(1-p)\theta} \\ 0 \\ 0 \\ u_p(r)\mathrm{e}^{i(3-p)\theta} \end{pmatrix}. \tag{4.3}$$

Here $p = 1, 2, 3$ enumerates three modes (three fermionic generations). The radial functions q_p are solutions to the following set of the equations,

$$\begin{cases} \partial_r q_p - \frac{(3-p)}{r} q_p + g_q v^3 F^3 q_{4-p} = 0 \\ \partial_r q_{4-p} - \frac{(p-1)}{r} q_{4-p} + g_q v^3 F^3 q_p = 0 \end{cases} \tag{4.4}$$

with the normalization condition

$$\int_0^\infty r dr (q_p^2 + q_{4-p}^2) = \frac{1}{2\pi}\,. \tag{4.5}$$

The functions $u_p(r)$ satisfy the same equations with the replacement $g_q \to g_u$.

Since the background radial function $F(r)$ is not known in analytical form, eqs. (4.4) can be solved only numerically. We will present the numerical results hereinafter, now, however, let us consider the equations in more details.

We are interested in obtaining low energy fermion masses which originate from the integration over extra dimensions of the following expression

$$Y_u \int d^2x \tilde{H} X^* \bar{Q} \frac{1-\Gamma_7}{2} U + \text{h.c.} \tag{4.6}$$

Substituting here the zero modes (4.3), and integrating over polar angle θ one finds

$$\begin{aligned} m^u_{ps} &= Y_u \int r dr d\theta H(r) X(r) q_p(g_q, r) u_s(g_u, r) \mathrm{e}^{i\theta(r-s)} \\ &= \delta_{ps} 2\pi Y_u \int_0^\infty r dr H(r) X(r) q_p(g_q, r) u_p(g_u, r)\,. \end{aligned} \tag{4.7}$$

It is worth noting that due to orthogonality of the zero modes and due to the fact that H and X do not depend on θ the integral over θ leads to a selection rule, $m^u_{ps} \sim \delta_{ps}$. To obtain non-trivial inter-generation mixings one should also consider the term

$$Y_u \epsilon_u \int d^2x \tilde{H} \Phi^* \bar{Q} \frac{1-\Gamma_7}{2} U + \text{h.c.} \tag{4.8}$$

The non-trivial θ-dependence of Φ (3.3) leads to the off-diagonal elements in the mass matrix

$$\begin{aligned} m^u_{ps} &= Y_u \epsilon_u \int r dr d\theta H(r) \Phi(r) q_p(g_q, r) u_s(g_u, r) \mathrm{e}^{i\theta(r-s-1)} \\ &= \delta_{p,s+1} 2\pi Y_u \epsilon_u \int_0^\infty r dr H(r) \Phi(r) q_p(g_q, r) u_{p-1}(g_u, r)\,. \end{aligned} \tag{4.9}$$

JHEP04(2002)055

In the same way one finds non-zero off-diagonal elements for down-type quarks $m^d_{ps} \sim \delta_{p+1,s}$. Therefore, the mass matrices have the following form

$$m^u = \begin{pmatrix} m_{11} & 0 & 0 \\ m_{21} & m_{22} & 0 \\ 0 & m_{32} & m_{33} \end{pmatrix}, \qquad m^d = \begin{pmatrix} m_{11} & m_{12} & 0 \\ & m_{22} & m_{23} \\ 0 & 0 & m_{33} \end{pmatrix}.$$

To evaluate the integrals (4.7) and (4.9) over r we take into account that the functions $H(r)$ and $X(r)$ are localized inside the vortex. So, the integral (4.7) is saturated near the origin, and therefore only the behavior of the zero modes at $r \to 0$ is relevant. It is easy to find from equations (4.4) that $q_p \sim r^{3-p}$ at $r \to 0$, however, it is not so easy to obtain the dependence on g_q of the zero modes. To obtain the dependence on small Yukawa couplings g of the fermion masses we should take into account the normalization condition (4.5) and investigate the behavior of the zero modes near the origin more carefully. To do this let us consider the following simplified model.

Let us approximate the vortex background by the functions

$$\begin{aligned} v^3 F(r)^3 &= \begin{cases} r^3\,, & r < 1 \\ 1\,, & r \geq 1 \end{cases} \\ H(r) = X(r) &= \begin{cases} 1\,, & r < 1 \\ 0\,, & r \geq 1 \end{cases} \end{aligned} \tag{4.10}$$

These functions have the correct behavior at the origin and at infinity. So, this is a reasonable approximation. The zero mode equations (4.4) with the background eq. (4.10) have analytical solution in terms of modified Bessel functions. However, to clarify the picture, we present here only the leading dependence on g of the solutions at $r < 1$ (which is only relevant for our purposes), and exact solutions at $r \geq 1$:

$$\begin{aligned} q_1(r) &= \begin{cases} \tilde{C}_1 r^2\,, & r < 1 \\ \frac{Cr}{\sqrt{g_q}} e^{-g_q r}\,, & r \geq 1 \end{cases} \qquad q_2(r) = \begin{cases} \tilde{C}_2 r\,, & r < 1 \\ \frac{C_m r}{\sqrt{g_q}} e^{-g_q r}\,, & r \geq 1 \end{cases} \\ q_3(r) &= \begin{cases} \tilde{C}_3\,, & r < 1 \\ \frac{Cr}{\sqrt{g_q}} e^{-g_q r}(1 + \frac{1}{g_q r})\,, & r \geq 1 \end{cases} \end{aligned} \tag{4.11}$$

To find the dependence on g_q of the coefficients C and $\tilde{C}$ let us assume that the normalization integral (4.5) is saturated at $r \geq 1$. Then substituting the solutions (4.11) into the condition (4.5) one has $C \simeq C_m \simeq g_q^{5/2}$. Thus, from the continuity of the solutions at $r = 1$, we find that $\tilde{C}_{1,2} \simeq g_q^2$ and $\tilde{C}_3 \simeq g_q$. This means in particular that our assumption is indeed valid: the normalization integrals at $r < 1$ are of order g_q^2 and can be neglected.

Now we can find the mass pattern in this simplified model. Substituting the solutions (4.11) and background (4.10) into integral (4.7) we have the following dependence on $g_{q,u}$ of the fermion masses,

$$m_{11} \sim m_{22} \sim (g_q g_u)^2\,, \qquad m_{33} \sim (g_q g_u)\,. \tag{4.12}$$

We see that m_{11} and m_{22} are parametrically of the same order, and therefore there is no hierarchy in the case of global $\mathrm{U}_g(1)$. This result was confirmed by the numerical evaluation of the zero modes and the integral (4.7).

4.2 Gauged $\mathrm{U}_g(1)$

Let us concentrate now on the case of the gauged $\mathrm{U}_g(1)$. In this case the equations for the zero modes can be obtained from (4.4) by the replacement

$$\partial_r \to \partial_r + \frac{3A(r)}{2r}\,.$$

Therefore, all dependence of the zero modes on the gauge field can be absorbed into a factor

$$(q_p)_{\rm gauge} = \exp\left(-\int^r \frac{3A(r')}{2r'}dr'\right)\cdot (q_p)_{\rm global}\,. \tag{4.13}$$

Approximating the gauge field as

$$A(r) = \begin{cases} 0\,, & r<1 \\ 1\,, & r\geq 1 \end{cases} \tag{4.14}$$

we find the following zero modes

$$q_1(r) \;=\; \begin{cases} \tilde{C}_1 r^2\,, & r<1 \\ \frac{C}{\sqrt{g_q r}} e^{-g_q r}\,, & r\geq 1 \end{cases} \qquad q_2(r) = \begin{cases} \tilde{C}_2 r\,, & r<1 \\ \frac{C_m}{\sqrt{g_q r}} e^{-g_q r}\,, & r\geq 1 \end{cases}$$
$$q_3(r) \;=\; \begin{cases} \tilde{C}_3\,, & r<1 \\ \frac{C}{\sqrt{g_q r}} e^{-g_q r}(1+\frac{1}{g_q r})\,, & r\geq 1 \end{cases} \tag{4.15}$$

The situation changes drastically compared to the global $\mathrm{U}_g(1)$ group. The continuity at $r=1$ results in the following relations between $\tilde{C}_p$ and C, C_m:

$$\tilde{C}_1 = C g_q^{-1/2}\,, \qquad \tilde{C}_2 = C_m g_q^{-1/2}\,, \qquad \tilde{C}_3 = C g_q^{-3/2}\,. \tag{4.16}$$

Using the normalization condition (4.5) one obtains that the main contribution to the integral (4.5) for the second mode comes from $r>1$, and, so, $C_m \sim g_q$. However, the normalization integral for the first and third modes is saturated now in the core $r<1$ by the third mode. Thus, $C \sim g_q^{3/2}$ and

$$\tilde{C}_p \sim g_q^{\frac{3-p}{2}}\,. \tag{4.17}$$

With the dependence (4.17), one finds the hierarchical mass pattern in the case of gauge vortex

$$m^u_{pp} \sim Y_u (g_u g_q)^{\frac{3-p}{2}} \tag{4.18}$$

which was announced in the introduction.

In the same way one estimates mixing terms from the lagrangian (2.4), (4.8)

$$m^u_{ps} \sim \delta_{p,s+1} Y_u \epsilon_u \sqrt{g_u} (g_u g_q)^{\frac{3-p}{2}}\,, \tag{4.19}$$

and

$$m^u \sim Y_u \begin{pmatrix} g_u g_q & 0 & 0 \\ \epsilon_u g_u \sqrt{g_q} & \sqrt{g_u g_q} & 0 \\ 0 & \epsilon_u \sqrt{g_u} & 1 \end{pmatrix}. \tag{4.20}$$

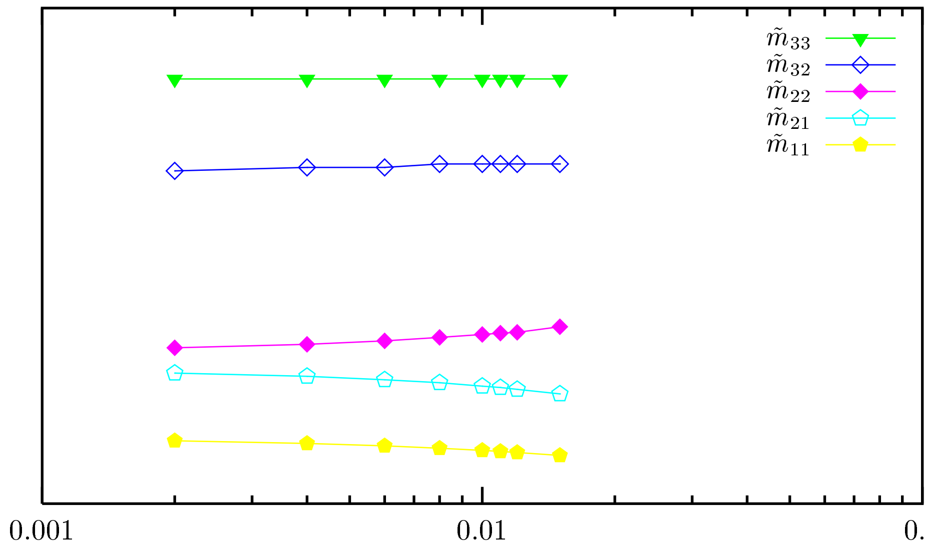

Figure 2: The g-dependence of the fermion mass pattern $\tilde{m}_{ps} = m_{ps}/g^{(6-p-s)/2}$ calculated in the background shown at figure 1.

Of course, the vortex background obtained numerically differs from our crude approximation. In particular, one can see from figure 1 that the gauge field $A(r)$ is wider than the scalar $F(r)$, and even far outside the core is almost equal to zero. So, our approximation (4.14) is not quite appropriate. One may expect that in the real background (figure 1) the g-dependence of the mass pattern is more resembling with the dependence in the case of the global $\mathrm{U}_g(1)$ group (4.12) when $A(r) = 0$. However, the numerical calculations show that the mass pattern depends weakly on the width of the gauge field. We present in figure 2 five non-zero elements of the mass matrix (4.20) divided by the corresponding power of g in the case $g = g_q = g_u$. We see that $m_{ps}/g^{(6-p-s)/2}$ indeed depend weakly on g in the wide interesting range.

5. Results for Standard Model fermions

Now we are ready to find the set of the parameters of the our model (Y, ϵ, g) which reproduce the real SM fermionic mass pattern.

To do this, one has to find eigenvalues of the mass matrix $m^u_{ps}(Y_u, \epsilon_u, g_q, g_u)$, to equate the result with known fermion masses, and to obtain three equations on four unknown variable, Y_u, ϵ_u, g_q, g_u. However, even to solve numerically these equations is not an easy problem. The point is that to find the solution one has to obtain the mass matrix m^u_{ps} in some range of the parameters, which requires in turn the calculation of the zero modes (solving the eqs. (4.4)) at each value of g. Fortunately, there is a simpler way. First of all, note that the mass matrix is proportional to $Y_{u,(d,l)}$, and, therefore, any ratio of the masses does not depend on $Y_{u,(d,l)}$. Forming two ratios of the masses one obtains two equations on three variables ϵ_u, g_q, g_u. The second simplification is to use the established dependence of the mass matrix on g and ϵ (4.18), (4.19). After this the obtained equations can be easily solved. In fact, in the quark sector the situation is more complicated since besides the mass matrix for up-quarks, there are also the mass matrix for down quarks and the CKM matrix. Therefore, we have seven equations (four equations for the mass ratios and three equations for CKM angles) on five parameters (g_q, g_u, g_d, ϵ_u, ϵ_d). In our simulations we have used five of seven equations to reproduce correct mass ratios and one element of the CKM-matrix (U_{33}^{CKM}). As a result we have obtained

$$g_q = 0.0013 \cdot M^{-5}\,, \quad g_u = 0.0013 \cdot M^{-5}\,, \quad g_d = 0.03 \cdot M^{-5}\,, \quad \epsilon_u = 0.9\,, \quad \epsilon_d = 0.1\,. \tag{5.1}$$

Having at hand these parameters, one can find two other independent elements of the CKM-matrix, but, in general, we could not expect that they will coincide with the known values. However, we have obtained the following CKM-matrix

$$U^{CKM} = \begin{pmatrix} 0.9780 & 0.2084 & 0.0100 \\ 0.2086 & 0.9769 & 0.0453 \\ 0.0004 & 0.0464 & 0.9990 \end{pmatrix}, \tag{5.2}$$

and see that this matrix coincides quite well with the measured CKM-matrix in the Standard Model [9]

$$U_{SM}^{CKM} = \begin{pmatrix} 0.9742 \div 0.9757 & 0.219 \div 0.226 & 0.002 \div 0.005 \\ 0.219 \div 0.225 & 0.9734 \div 0.9749 & 0.037 \div 0.043 \\ 0.004 \div 0.014 & 0.035 \div 0.043 & 0.9990 \div 0.9993 \end{pmatrix}. \tag{5.3}$$

In the leptonic sector the correct mass ratios are reproduced at

$$g_l = g_e = 0.01\,, \qquad \epsilon_l = 0.75\,.$$

To find Yukawa couplings Y we should first of all understand how the obtained dimensionful parameters (that is all parameters except ϵ) correlate with the absolute energy scale. Let us assume that the first non-zero fermion level is of order 100 TeV. It means that $gv^3 \simeq 100$ TeV. Substituting here $g = 0.001$ and $v = 1$ one finds our units:

$$1 \cdot M = 10^5\,\mathrm{TeV}\,.$$

With this relation, one can convert all parameters measured in our units to the absolute mass scale:

$$\begin{aligned} v = \mu &= 10^{10}\,\mathrm{TeV}^2\,, & v_1 &= 3 \cdot 10^9\,\mathrm{TeV}^2\,, \\ \lambda = \kappa = \rho &= 10^{-10}\,\mathrm{TeV}^{-2}\,, & h &= 1.02 \cdot 10^{-5}\,\mathrm{TeV}^{-1}\,, \\ \eta &= 3.1 \cdot 10^{-6}\,\mathrm{TeV}^{-1}\,, & e &= 4 \cdot 10^{-7}\,\mathrm{TeV}^{-1}\,, \\ g_q = g_u &= 1.3 \cdot 10^{-28}\,\mathrm{TeV}^{-5}\,, & g_d &= 3 \cdot 10^{-27}\,\mathrm{TeV}^{-5}\,, \qquad g_l = g_e = 1 \cdot 10^{-27}\,\mathrm{TeV}^{-5}\,. \end{aligned}$$

Now taking into account the actual values of the fermion masses one finds the values of the Yukawa couplings Y:

$$Y_u = 3.7 \cdot 10^{-17}\,\mathrm{TeV}^{-3}\,, \qquad Y_d = 2.1 \cdot 10^{-20}\,\mathrm{TeV}^{-3}\,, \qquad Y_l = 1.8 \cdot 10^{-21}\,\mathrm{TeV}^{-3}\,.$$

It is worth noting that we do not obtain any additional hierarchy: all dimensionless combinations of the our parameters are of order unity. For instance, $g_u^{1/5}/\sqrt{\lambda} = 0.4$, $Y_u^{1/3}/g_u^{1/5} = 1.3$, etc.

JHEP04(2002)055

6. Conclusions

In a class of multidimensional models with one vector-like fermionic family, the low-energy effective theory describes three chiral families in four dimensions. Hierarchy of fermionic masses appears due to different profiles of the fermionic wave functions in extra dimensions. In this paper we have investigated numerically one of the models of this kind. Namely, we considered the model suggested in [2] and [3]. We have shown that hierarchical mass pattern indeed appears in the model [3] with the gauged $\mathrm{U}_g(1)$ group which is responsible for the forming of Abrikosov-Nielsen-Olesen vortex, but the dependence of the mass matrix on the parameters is slightly different than it was presented in the refs. [2, 3]. On the other hand, in the case of global $\mathrm{U}_g(1)$ unnatural mass pattern appears. In the gauge case we found the set of the parameters of the model which reproduces well all known fermion masses and mixings, without hierarchy among parameters. Moreover, in SM quark sector we have reproduced the nine known values (six masses and three mixing angles) by the variation of the seven parameters of the model: three couplings $g_{u,d,q}$, two $\epsilon_{u,d}$, and two Yukawa couplings $Y_{u,d}$.

Acknowledgments

We are indebted to V. Rubakov, S. Troitsky, and J.-M. Frère for numerous helpful discussions. We thank F. Bezrukov and A. Kuznetsov for the help in numerical simulations. This work is supported in part by RFFI grants 99-02-18410a, 01-02-06033, by CRDF award RP1-2103, by the Council of Presidential Grants and State Support of Leading Scientific Schools, grant 00-15-96626, and by the programme SCOPES of the Swiss National Science Foundation, project No. 7SUPJ062239, financed by Federal Department of Foreign affairs.

References

[1] N. Arkani-Hamed and M. Schmaltz, *Hierarchies without symmetries from extra dimensions*, *Phys. Rev.* **D 61** (2000) 033005 [hep-ph/9903417];
G.R. Dvali and M.A. Shifman, *Families as neighbors in extra dimension*, *Phys. Lett.* **B 475** (2000) 295 [hep-ph/0001072];
T. Gherghetta and A. Pomarol, *Bulk fields and supersymmetry in a slice of AdS*, *Nucl. Phys.* **B 586** (2000) 141 [hep-ph/0003129];
D.E. Kaplan and T.M.P. Tait, *Supersymmetry breaking, fermion masses and a small extra dimension*, *J. High Energy Phys.* **06** (2000) 020 [hep-ph/0004200];
S.J. Huber and Q. Shafi, *Fermion masses, mixings and proton decay in a Randall-Sundrum model*, *Phys. Lett.* **B 498** (2001) 256 [hep-ph/0010195].

[2] M.V. Libanov and S.V. Troitsky, *Three fermionic generations on a topological defect in extra dimensions*, *Nucl. Phys.* **B 599** (2001) 319 [hep-ph/0011095].

[3] J.M. Frère, M.V. Libanov and S.V. Troitsky, *Three generations on a local vortex in extra dimensions*, *Phys. Lett.* **B 512** (2001) 169 [hep-ph/0012306].

[4] V.A. Rubakov, *Large and infinite extra dimensions: an introduction*, *Phys. Usp.* **44** (2001) 871 [hep-ph/0104152].

[5] J.M. Frère, M.V. Libanov and S.V. Troitsky, *Neutrino masses with a single generation in the bulk*, *J. High Energy Phys.* **11** (2001) 025 [hep-ph/0110045].

[6] E. Witten, *Superconducting strings*, *Nucl. Phys.* **B 249** (1985) 557.

[7] W.H. Press et al., *Numerical recipies in C*, Cambridge University Press, Cambridge UK 1992.

[8] R. Jackiw and P. Rossi, *Zero modes of the vortex-fermion system*, *Nucl. Phys.* **B 190** (1981) 681;
E.J. Weinberg, *Index calculations for the fermion-vortex system*, *Phys. Rev.* **D 24** (1981) 2669.

[9] F.J. Gilman, K. Kleinknecht and B. Renk, *The Cabibbo-Kobayashi-Maskawa quark mixing matrix: in review of particle physics (RPP 2000)*, *Eur. Phys. J.* **C 15** (2000) 110.

JHEP04(2002)055